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Causal Green Function in Relativistic Quantum Mechanics

Egon Marx[1]

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The causal Green function or Feynman propagator for the free-field Klein-Gordon equation and related singular functions, defined as distributions, are related to the causal time-boundary data. Probability densities and amplitudes are defined in terms of the solutions of the Klein-Gordon equation for a complex scalar field interacting with an electromagnetic field. The convergence of the perturbation expansion of the solution of the Klein-Gordon equation for a charged scalar particle in an external field is shown for well-behaved electromagnetic potentials. Other relativistic wave equations are discussed briefly.

1. INTRODUCTION

As the nonrelativistic theory of quantum mechanics was being developed, physicists were well aware that this could only be an approximation to the correct theory, which had to be invariant under Lorentz transformations. This was especially true for the interaction of charged particles with the electromagnetic field, since Maxwell's equations were already relativistically covariant.

The two principal relativistic generalizations of the Schrödinger equation, the Klein-Gordon equation and the Dirac equation, were only partially successful and were beset by difficulties, such as negative-energy states, indefinite probability densities, and *Zitterbewegung.* Relativistic quantum mechanics (RQM) (Dirac, 1932) was superseded by quantum field theory (QFT), which was found to be better suited to describe the creation and annihilation of particles. The standard forms of QFT are subject to other difficulties, mainly problems with divergent terms in the perturbation

[1]National Bureau of Standards, Gaithersburg, Maryland 20899.

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expansion of the transition amplitudes or related quantities. These problems have been removed by means of the renormalization procedure. There is no clear distinction between the standard formulations of RQM (Bjorken and Drell, 1964) and QFT (Bjorken and Drell, 1965), since both are made to yield the same set of Feynman diagrams for a perturbation expansion.

We have developed a different version of RQM for a scalar particle in an external electromagnetic field (Marx, 1969, 1970a). This theory allows for pair creation and annihilation, as well as particle and antiparticle scattering, while retaining a classical field as the wave function that represents the particles. This is accomplished by considering the antiparticle as a particle propagating backward in time (Stueckelberg, 1941, 1942; Feynman, 1949). Thus, when we observe pair annihilation, the antiparticle is the same physical entity as the particle which was turned around in time by the interaction. This theory can be generalized to several identical particles (Marx, 1970a) in a many-times formalism. The extension to spin-1/2 particles requires a modification of the Dirac equation (Marx, 1970b) similar to that introduced in QFT by the anticommutation of operators, and this theory can be recast into a form of QFT (Marx, 1972) with a fixed number of particles.

We have separated the relativistic wave functions into positive- and negative-frequency parts, although these concepts apply in a strict sense only when there is no interaction. We have defined probability amplitudes for particles and antiparticles, taking advantage of the conservation of electrical charge. The Green function used in RQM (Bjorken and Drell, 1964; Marx, 1969) is neither the advanced nor the retarded one, but the Feynman propagator (Feynman, 1949) or causal Green function (Stueckelberg and Rivier, 1950; Fierz, 1950). The time-boundary data that are required to write a solution of the field equation of motion in terms of this Green function are the positive-frequency part of the wave function at the initial time and the negative-frequency part at the final time.

Here we present a rigorous definition of the probability amplitudes in terms of the wave function within the context of the theory of distributions and exhibit the connection between the causal Green function and the time-boundary conditions. The electromagnetic potentials have to be more restricted where they multiply a distribution than when they multiply a function. To define current densities and probability densities we assume that the wave functions are square-integrable functions of the space variables. We find the perturbation expansion of a solution of the Klein–Gordon equation for a charged scalar particle in a given electromagnetic field and we show the convergence of the series.

The causal Green function for the Klein–Gordon equation is customarily (Schweber, 1961; Jauch and Rohrlich, 1976) defined as a function by

$$\Delta_F(x) = \lim_{\varepsilon \to 0+} \frac{1}{(2\pi)^4} \int \frac{d^4k \exp(-ik \cdot x)}{k^2 - m^2 + i\varepsilon} \tag{1}$$

where x and k are four-vectors, $x = (t, \mathbf{x})$, $k = (k_0, \mathbf{k})$, and m is the mass of the particle. This function (actually a distribution) satisfies

$$(\partial^2 + m^2)\Delta_F(x) = -\delta(x) \tag{2}$$

where $\delta(x)$ is the four-dimensional Dirac delta function and $\partial_\mu = \partial/\partial x^\mu$. The Green function Δ_F can be expressed in terms of positive- and negative-frequency solutions of the homogeneous Klein–Gordon equation, $\Delta^{(\pm)}$, by

$$\Delta_F(x) = \Delta^{(+)}(x)\theta(t) - \Delta^{(-)}(x)\theta(-t) \tag{3}$$

where

$$\Delta^{(\pm)}(x) = \mp i(2\pi)^{-3} \int d^3k\, (2k_0)^{-1} \exp(\mp ik \cdot x) \tag{4}$$

and

$$k_0 = +(\mathbf{k}^2 + m^2)^{1/2} \tag{5}$$

and θ is the unit step function.

We give proper definitions (Gel'fand and Shilov, 1964) of these and related distributions in Section 2, and we use these distributions to write the solution of the Klein–Gordon equation with a given source and appropriate time-boundary conditions. We examine briefly the extension of this approach to the d'Alembert equation, the Dirac equation, and the Weyl equation in Section 3. In Section 4 we elaborate on the probabilistic interpretation of the wave function and compare this interpretation to standard formulations of RQM and QFT. In Section 5 we determine the perturbation expansion of the wave function of a scalar particle in an electromagnetic field and we prove the convergence of this series. We discuss some of the difficulties encountered in the RQM of a spin-1/2 particle in Section 6. We use natural units ($c = 1$, $\hbar = 1$) and the time-favoring metric in space-time. Greek indices for four-vectors range from 0 to 3 and we use the modified summation convention for repeated lower Greek indices,

$$a \cdot b = a_\mu b_\mu = a_0 b_0 - \mathbf{a} \cdot \mathbf{b} \tag{6}$$

We follow the customary notation for distributions and their Fourier transforms (Schwartz, 1966; Stackgold, 1979; Marx and Maystre, 1982).

2. THE CAUSAL GREEN FUNCTION FOR THE KLEIN-GORDON EQUATION

In this section we define the causal Green function and related distributions that we need to establish the probabilistic interpretation of the relativistic wave function and to find the perturbation expansion of the solution of the Klein-Gordon equation. We define projection operators for the free-field Klein-Gordon equation, and we relate the causal Green function to the causal time-boundary conditions.

The $\Delta^{(\pm)}$ are solutions of the homogeneous Klein-Gordon equation; they are elements of the space $\mathscr{S}'$ of tempered distributions defined as linear functionals on the space $\mathscr{S}$ of infinitely differentiable complex functions of four real variables that decrease rapidly at infinity. The informal definition (4) is replaced by

$$\langle \Delta^{(\pm)}, f\rangle = \mp i(2\pi)^{-3/2} \int_{-\infty}^{\infty} dt \int d^3k \times (2k_0)^{-1} \tilde{f}_{\pm}(\mathbf{k}, t) \exp(\mp i k_0 t), \qquad f \in \mathscr{S} \tag{7}$$

where k_0 is given by (5), and we use the tilde to indicate the Fourier transform of the test function f,

$$\tilde{f}_{\pm}(\mathbf{k}, t) = (2\pi)^{-3/2} \int d^3x f(\mathbf{x}, t) \exp(\pm i\mathbf{k}\cdot\mathbf{x}) = \mathscr{F}_{\pm}[f(\mathbf{x}, t)] \tag{8}$$

The definition of the distribution Δ_F in (3) is replaced by

$$\langle \Delta_F, f\rangle = -i(2\pi)^{-3/2} \int d^3k\, (2k_0)^{-1} \left[\int_0^{\infty} dt\, \tilde{f}_{+}(\mathbf{k}, t) \exp(-ik_0 t) + \int_{-\infty}^{0} dt\, \tilde{f}_{-}(\mathbf{k}, t) \exp(ik_0 t) \right] \tag{9}$$

We define the operators (Marx, 1969) that separate the positive- and negative-frequency parts of tempered distributions,

$$P_0^{(\pm)} = \tfrac{1}{2}(1 \pm i\tilde{E}^{-1}\partial_0) \tag{10}$$

where a function of $\tilde{E} = (-\nabla^2 + m^2)^{1/2}$ is defined by

$$\langle F(\tilde{E})\phi, f\rangle = \langle F(k_0)\tilde{\phi}, \tilde{f}\rangle = \langle \tilde{\phi}, F(k_0)\tilde{f}\rangle \tag{11}$$

for an arbitrary tempered distribution $\phi \in \mathscr{S}'$. These operators will be modified slightly when an electromagnetic field is present. Note that we do not transform the time variable and that, as long as m is not equal to zero,

$F(k_0)$ is an infinitely differentiable function of the k_i that does not change the behavior of f at infinity for the functions F that we use in this paper. We define the positive- and negative-frequency parts of a distribution ϕ by

$$\phi^{(\pm)} = P_0^{(\pm)}\phi \tag{12}$$

The operators $P_0^{(\pm)}$ satisfy

$$P_0^{(+)} + P_0^{(-)} = 1 \tag{13}$$

$$P_0^{(+)}P_0^{(-)} = P_0^{(-)}P_0^{(+)} = (2\tilde{E})^{-2}(\partial^2 + m^2) \tag{14}$$

$$(P_0^{(\pm)})^2 = P_0^{(\pm)} - (2\tilde{E})^{-2}(\partial^2 + m^2) \tag{15}$$

whence the $P_0^{(\pm)}$ are projection operators in the space of solutions of the homogeneous Klein–Gordon equation, although ϕ is not restricted to this subspace.

We now use the causal Green function to solve the inhomogeneous Klein–Gordon equation

$$\{(\partial^2 + m^2)\phi(x)\} = \omega(x), \qquad t_i < t < t_f \tag{16}$$

where the braces indicate that the derivatives are taken in the sense of functions, that is, we are assuming that ϕ is a twice differentiable function of the x_μ in this region.

We use operators $P_0^{(\pm)}$ to find the positive- and negative-frequency parts of Δ_F as defined in (9) and obtain

$$\langle \Delta_F^{(+)}, f\rangle = -i(2\pi)^{-3/2}\int d^3k\,(2k_0)^{-1} \times \int_0^\infty dt\,\tilde{f}_+(\mathbf{k}, t)\exp(-ik_0 t) \tag{17}$$

$$\langle \Delta_F^{(-)}, f\rangle = -i(2\pi)^{-3/2}\int d^3k\,(2k_0)^{-1} \times \int_{-\infty}^0 dt\,\tilde{f}_-(\mathbf{k}, t)\exp(ik_0 t) \tag{18}$$

which are similar but not equivalent to the definitions of $\Delta^{(\pm)}$ in (7).

To allow for the inclusion of time-boundary conditions, we assume that the distributions ϕ and $\dot{\phi}$ are functions with jump discontinuities $\Delta\phi_i$ and $\Delta\dot{\phi}_i$ at t_i, and $\Delta\phi_f$ and $\Delta\dot{\phi}_f$ at t_f. Equation (16) then is extended to

$$(\partial^2 + m^2)\phi = \omega + \Delta\phi_i(\mathbf{x})\,\delta'(t - t_i) + \Delta\dot{\phi}_i(\mathbf{x})\,\delta(t - t_i) + \Delta\phi_f(\mathbf{x})\,\delta'(t - t_f) + \Delta\dot{\phi}_f(\mathbf{x})\,\delta(t - t_f) = \check{\omega} \tag{19}$$

which defines the source distribution $\check{\omega}$. Then

$$\phi = -\Delta_F * \check{\omega} \tag{20}$$

$$\phi^{(\pm)} = -\Delta_F^{(\pm)} * \check{\omega} \tag{21}$$

where the asterisk indicates the convolution product. We can specify arbitrarily the *jumps* in the function ϕ and its time derivative at t_i and t_f, but they will not correspond to boundary *values* unless further restrictions are imposed on the solution outside the interval.

We first recall the solution of the Klein-Gordon equation by means of the retarded Green function Δ_R, which also satisfies (2) and vanishes for $t<0$. The solution is still given by (20) where Δ_F is replaced by Δ_R. We assume that ϕ vanishes for $t<t_i$ and we give the initial values of ϕ and $\dot{\phi}$, which are then equal to the jumps at t_i. This solution satisfies the Klein-Gordon equation and the initial value assignments; we disregard the jumps at t_f, which affect the solution only for $t>t_f$. If final values were given, we could still use the retarded Green function, but the boundary conditions would have to be satisfied via integral equations; it would be much easier to use the advanced Green function.

We have to examine the solution further to find the natural boundary conditions for the causal Green function, which allow us to find the solution directly from (20) or (21). The jumps in $\phi^{(\pm)}$ can be expressed in terms of the jumps in ϕ and $\dot{\phi}$ through (12). Equations (17) and (18) show that $\Delta_F^{(+)}$ vanishes for $t<0$ and that $\Delta_F^{(-)}$ vanishes for $t>0$. We use (10), (14), and (15) to derive

$$\begin{aligned}\Delta_F^{(+)} * [\Delta\phi_i(\mathbf{x})\, \delta'(t-t_i)) + \Delta\dot{\phi}_i(\mathbf{x})\, \delta(t-t_i)]\\ = -2i\Delta_F^{(+)}(\mathbf{x}, t-t_i) * \tilde{E}\Delta\phi_i^{(+)}(\mathbf{x}) - \tfrac{1}{2}i\tilde{E}^{-1}\Delta\phi_i(\mathbf{x})\, \delta(t-t_i)\end{aligned} \tag{22}$$

where the convolution on the right side extends over the three-dimensional $\mathbf{x}$ space only and the second term does not affect the value of the function for $t>t_i$. Equation (22) and similar relations at the final time and for the negative-frequency part show that $\phi^{(+)}$ is determined for times between t_i and t_f by its discontinuity at t_i and the source ω, and that $\phi^{(-)}$ is determined in this time interval by its discontinuity at t_f and the source ω. We consequently assume that $\phi^{(+)}$ vanishes for $t<t_i$ and has a jump equal to $\phi_i^{(+)}(\mathbf{x})$ at $t=t_i$, and that $\phi^{(-)}$ vanishes for $t>t_f$ and has a jump equal to $-\phi_f^{(-)}(\mathbf{x})$ at $t=t_f$. We set

$$\phi^{(+)} = -\Delta_F^{(+)} * [\omega(x) - 2i\tilde{E}\phi_i^{(+)}(\mathbf{x})\, \delta(t-t_i)] \tag{23}$$

$$\phi^{(-)} = -\Delta_F^{(-)} * [\omega(x) + 2i\tilde{E}\phi_f^{(-)}(\mathbf{x})\, \delta(t-t_f)] \tag{24}$$

where ω is known for $t_i<t<t_f$ and we assume that it vanishes outside this interval. Thus, $\phi = \phi^{(+)} + \phi^{(-)}$ is determined by the source ω, the value of

$\phi^{(+)}$ at t_i, and the value of $\phi^{(-)}$ at t_f, which are the causal time-boundary conditions.

We compute

$$(\partial^2+m^2)\phi=\omega+\phi_i^{(+)}\,\delta'(t-t_i)-i\tilde{E}\phi_i^{(+)}\,\delta(t-t_i)$$
$$-\phi_f^{(-)}\,\delta'(t-t_f)+i\tilde{E}\phi_f^{(-)}\,\delta(t-t_f) \qquad (25)$$

and compare this equation with (19) to determine the jumps in ϕ and $\dot{\phi}$ at times t_i and t_f in terms of the given boundary values.

In Section 4 we show how these time-boundary conditions are related to the probabilistic interpretation of the relativistic wave function.

3. CAUSAL GREEN FUNCTIONS FOR OTHER EQUATIONS

For $m=0$, the Klein–Gordon equation reduces to the d'Alembert equation. Instead of (5), we have

$$k_0=(k_1^2+k_2^2+k_3^2)^{1/2} \qquad (26)$$

and k_0^ν is no longer an infinitely differentiable function of the k_i for arbitrary ν. Since

$$d^3k=k_0^2\,dk_0\,d\Omega_{\mathbf{k}} \qquad (27)$$

where $d\Omega$ is the element of solid angle, the definitions (7) for $\Delta^{(\pm)}$ and (9) for Δ_F give the correct definitions for $D^{(\pm)}$ and D_F in the limit $m\to 0$. Nevertheless, we cannot use the causal Green function for a real field (such as the electromagnetic fields and potentials), because the positive-frequency part of a real field determines the negative-frequency part and *vice versa.*

The Dirac equation for bispinors ψ is

$$(-i\gamma\cdot\partial+m)\psi(x)=\omega(x) \qquad (28)$$

where the γ_μ are a set of Dirac matrices that satisfy

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu} \qquad (29)$$

and ω is now a spinor source. The causal Green function for this equation is

$$S_F=(i\gamma\cdot\partial+m)\Delta_F \qquad (30)$$

The projection operators (10) are not useful for a bispinor field because, for a first-order equation of motion, $\dot{\psi}$ is not given, but is determined by the equation. We define instead

$$P_0^{(\pm)}=\tfrac{1}{2}[1\pm\tilde{E}^{-1}(-i\boldsymbol{\alpha}\cdot\nabla+m\gamma_0)] \qquad (31)$$

where $\boldsymbol{\alpha}=\gamma_0\boldsymbol{\gamma}$; these are true projection operators that satisfy

$$(P_0^{(\pm)})^2=P_0^{(\pm)},\qquad P_0^{(+)}P_0^{(-)}=P_0^{(-)}P_0^{(+)}=0 \qquad (32)$$

for an arbitrary distribution ψ. These operators separate the two parts in the usual expansion of the spinor field,

$$\psi(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p\left(\frac{m}{E}\right)^{1/2}\sum_{\lambda=\pm1}[u_\lambda(\mathbf{p})b_\lambda(\mathbf{p},t)\exp(i\mathbf{p}\cdot\mathbf{x})$$
$$+v_\lambda(\mathbf{p})d_\lambda(\mathbf{p},t)\exp(-i\mathbf{p}\cdot\mathbf{x})] \tag{33}$$

where the index λ refers to the two helicity states and

$$E=p_0=(\mathbf{p}^2+m^2)^{1/2}$$
$$(p\cdot\gamma-m)u_\lambda(\mathbf{p})=0,\qquad (p\cdot\gamma+m)v_\lambda(\mathbf{p})=0 \tag{34}$$

The Weyl equation for a massless (two-component) spinor field χ is

$$-i\sigma_\mu^{\dot{A}B}\chi_{B,\mu}=\omega^{\dot{A}} \tag{35}$$

where the σ_μ are the unit matrix and a set of Pauli matrices which satisfy

$$\sigma_{\mu\dot{A}B}\sigma_\nu^{\dot{A}C}+\sigma_{\nu\dot{A}B}\sigma_\mu^{\dot{A}C}=2g_{\mu\nu}\delta_B{}^C \tag{36}$$

The corresponding causal Green function is

$$S_{F\dot{A}B}=i\sigma_{\mu\dot{A}B}D_{F,\mu} \tag{37}$$

Equation (35) is also a first-order equation, and there are only two independent amplitudes in χ. The solutions of the homogeneous equation have only one helicity state for each sign of the frequency; hence, we write the general expansion in the form

$$\chi(x)=(2\pi)^{-3/2}\int d^3p\,[\chi^{+1}(\hat{p})b(\mathbf{p},t)\exp(i\mathbf{p}\cdot\mathbf{x})$$
$$+\chi^{-1}(-\hat{p})d(\mathbf{p},t)\exp(-i\mathbf{p}\cdot\mathbf{x})] \tag{38}$$

where the helicity spinors satisfy

$$\boldsymbol{\sigma}\cdot\hat{p}\chi^\lambda(\hat{p})=\lambda\chi^\lambda(\hat{p}) \tag{39}$$

The corresponding projection operators are

$$P_0^{(\pm)}=\tfrac{1}{2}(1\mp i\tilde{E}^{-1}\boldsymbol{\sigma}\cdot\nabla) \tag{40}$$

which also satisfy (32).

4. PROBABILITY AMPLITUDES FOR SCALAR PARTICLES

We now show how we define probability amplitudes (Marx, 1969, 1970a) in a relativistic theory of the quantum mechanics of scalar particles in an external electromagnetic field. These amplitudes and the corresponding probability densities are a generalization of the basic concepts of the nonrelativistic theory. The probabilistic interpretation is closely related to the causal time-boundary conditions.

The equation of motion for the wave function ϕ is obtained from the Klein-Gordon equation by the gauge-invariant substitution

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu \tag{41}$$

which leads to the equation

$$(D^2 + m^2)\phi(x) = 0 \tag{42}$$

This is a second-order equation in time, which is equivalent to two Schrödinger-type equations. Thus, the wave function is related to two probability amplitudes $g^{(\pm)}(x)$, one for the particle and one for the antiparticle. To define the product of the distribution ϕ by the potentials A_μ, we restrict the potentials to be infinitely differentiable functions of slow growth at infinity.

We use the operators

$$P^{(\pm)} = \tfrac{1}{2}(1 \pm i\tilde{E}^{-1}D_0) \tag{43}$$

to define the positive- and negative-frequency parts of the wave function,

$$\phi^{(\pm)}(x) = P^{(\pm)}\phi(x) \tag{44}$$

and the probability amplitudes

$$g^{(\pm)}(x) = (2\tilde{E})^{1/2}\phi^{(\pm)}(x) \tag{45}$$

The definition (44) is not really covariant under simultaneous gauge transformations of the A_μ and ϕ, because $\exp[i\Lambda(x)]$ is affected by functions of the operator $\tilde{E}$, although the Klein-Gordon equation is covariant. We note that the A_0 could be made to vanish by choosing an appropriate gauge. We also usually assume that there is no overlap between the particle amplitudes and the external electromagnetic fields at t_i and t_f.

For the Klein-Gordon equation, the probability density of the nonrelativistic theory is replaced by the charge density j_0, which is part of the conserved current density

$$j_\mu = i[\phi^* D_\mu \phi - (D_\mu^* \phi^*)\phi] \tag{46}$$

In this equation and in what follows we interpret ϕ as the function that corresponds to the distribution. The functions are assumed to be square-integrable over all space for times between t_i and t_f even when multiplied by the potentials. This requirement relaxes the conditions on the potentials, and we find sufficient conditions for the convergence of the perturbation expansion in Section 5. The conserved charge is

$$Q = \int d^3x\, j_0(x) = A^{(+)}(t) - A^{(-)}(t) \tag{47}$$

where

$$A^{(\pm)}(t)=\int d^3x\,|g^{(\pm)}(x)|^2 \tag{48}$$

In particular, we interpret the $A^{(\pm)}$ at the initial and final times as probabilities. Conservation of charge implies that

$$A_f^{(+)}+A_i^{(-)}=A_i^{(+)}+A_f^{(-)} \tag{49}$$

where $A_{i,f}^{(\pm)}=A^{(\pm)}(t_{i,f})$. The quantities on the right side are determined by the given boundary values $\phi_i^{(+)}$ and $\phi_f^{(-)}$.

If we are describing particle scattering and pair annihilation, $g_f^{(-)}$ vanishes and $g_i^{(+)}$ is normalized to 1, that is,

$$A_i^{(+)}=1, \qquad A_f^{(-)}=0 \tag{50}$$

Equations (49) and (50) imply that

$$A_f^{(+)}+A_i^{(-)}=1, \qquad 0\le A_f^{(+)}\le 1, \qquad 0\le A_i^{(-)}\le 1 \tag{51}$$

and we interpret $A_f^{(+)}$ as the probability for particle scattering and $A_i^{(-)}$ as the probability for pair annihilation; one of the two has to occur.

If we are describing antiparticle scattering and pair annihilation, we similarly have

$$A_i^{(+)}=0, \qquad A_f^{(-)}=1 \tag{52}$$

and (51) is still valid. Now $A_f^{(+)}$ is the probability for pair creation and $A_i^{(-)}$ is the probability for antiparticle scattering. This selection of boundary conditions is reminiscent of the reflection and transmission of a particle by a potential barrier. Here the potential barrier occurs in the time direction; a particle coming from the past is either transmitted (particle scattering) or reflected (pair annihilation), *or* an antiparticle coming from the future is either transmitted (antiparticle scattering) or reflected (pair creation).

We interpret $|g^{(+)}(\mathbf{x}, t_f)|^2$ as the probability density for a particle at the final time and $|g^{(-)}(\mathbf{x}, t_i)|^2$ as the probability density for an antiparticle at the initial time. The quantities $A^{(\pm)}(t)$ at intermediate times are not restricted to be less than 1 and thus should not be interpreted in terms of probability amplitudes. This is not a serious restriction, since observations at these times would change the dynamical development of the system. Nevertheless, we extend the name of probability amplitudes to the $g^{(\pm)}$ and probability densities to the $\rho^{(\pm)}$ for arbitrary t, where

$$\rho^{(\pm)}(\mathbf{x}, t)=|g^{(\pm)}(\mathbf{x}, t)|^2 \tag{53}$$

We note that the charge density j_0 is not equal to the difference $\rho^{(+)}-\rho^{(-)}$ even when the electromagnetic field vanishes.

This definition of probability amplitudes is based on the work of Feshbach and Villars (1958), who also separate the solutions of the Klein-Gordon equation into their positive- and negative-frequency parts. In their approach, a wave function that is normalized so that the charge Q is $+1$ represents a particle. This implies that $A^{(+)}(t)$ is greater than 1 unless $\phi^{(-)}$ vanishes, which is incompatible with our probabilistic interpretation of the wave function. Similarly, a wave function for which Q is -1 represents an antiparticle, and this makes $A^{(-)}(t)$ greater than 1. Their approach is also used by Bjorken and Drell,[2] who define probability amplitudes in the case when the positive- and negative-frequency parts can be decoupled by a Foldy-Wouthuysen transformation.

Although these authors do not discuss the solution of the Klein-Gordon equation for particular boundary conditions, we see that their normalization of the charge can be effected by choosing the initial conditions for $g^{(+)}$ and $g^{(-)}$, or ϕ and $\dot{\phi}$, so that

$$A_i^{(+)} = 1, \qquad A_i^{(-)} = 0 \tag{54}$$

for particle scattering, or

$$A_i^{(+)} = 0, \qquad A_i^{(-)} = 1 \tag{55}$$

for antiparticle scattering. Even though this is a single-particle theory, the additional charges present at t_f might be interpreted in terms of the creation of one or more pairs, which is the picture that results from QFT.

Our probabilistic interpretation can be generalized to a system of n identical particles (Marx, 1970a). A symmetric wave function that depends on n four-vectors can be decomposed into $n+1$ independent amplitudes. The boundary conditions for a dynamical problem call for the specification of the $n+1$ amplitudes with all particles at the initial time and all antiparticles at the final time; one of these amplitudes is given normalized to 1 and the others are set equal to 0. The dynamics then determines the $n+1$ amplitudes with the particles at the final time and the antiparticles at the initial time, related to probability densities for these particles and antiparticles, and the different amplitudes correspond to mutually exclusive processes.

It is difficult to compare our theory with the standard QFT, mainly because there are many formulations of the latter that are equivalent to each other only in a more or less formal sense. Also, few of the formulations deal with the time translation operator $U(t, t_0)$; most of the results are formulated in terms of the scattering matrix S, which implies that $t_i \to -\infty$ and $t_f \to +\infty$. We note that even when the electromagnetic field is dynamical,

[2]See Bjorken and Drell (1964), pp. 198ff for a discussion of the interpretation of the scalar wave function.

our theory does not contain closed particle loops of other infinite terms such as particle self-energy contributions.

If we consider the standard QFT in the Schrödinger picture and specify a one-particle state at the initial time, the final state will have contributions from a one-particle state, a two-particle, one-antiparticle state, and so on. In our RQM, these added states only appear in other dynamical problems with two or more particle variables.

5. PROOF OF CONVERGENCE OF THE PERTURBATION EXPANSION

We now obtain the perturbation expansion (Marx, 1979) of the solution of (42) and prove its convergence for well-behaved potentials.

Here we are not interested in the convergence of a series of distributions, but we consider the corresponding functions and we prove convergence in the L^2 norm in space and L^∞ norm in the time interval (t_i, t_f).

We expand the wave function in a power series of the charge,

$$\phi(x) = \sum_{l=0}^{\infty} e^l \phi^{(l)}(x) \tag{56}$$

and the $\phi^{(l)}$ satisfy

$$(\partial^2 + m^2)\phi^{(0)} = 0 \tag{57}$$

$$(\partial^2 + m^2)\phi^{(1)} = -iA_{\mu,\mu}\phi^{(0)} - 2iA_\mu \phi^{(0)}_{,\mu} = \omega^{(1)} \tag{58}$$

$$(\partial^2 + m^2)\phi^{(l)} = -iA_{\mu,\mu}\phi^{(l-1)} - 2iA_\mu \phi^{(l-1)}_{,\mu} + A^2\phi^{(l-2)} = \omega^{(l)}, \qquad l \geq 2 \tag{59}$$

$$\phi_i^{(0)(+)} = \phi_i^{(+)}, \qquad \phi_f^{(0)(-)} = \phi_f^{(-)} \tag{60}$$

$$\phi_i^{(l)(+)} = 0, \qquad \phi_f^{(l)(-)} = 0, \qquad l \geq 1 \tag{61}$$

If the particle state is given at t_i by a square-integrable amplitude $\phi_i^{(+)}$ (the case of the antiparticle state given at t_f follows *mutatis mutandis*), the terms of the perturbation series are given by

$$\phi^{(0)} = 2i\Delta_F^{(+)} * \tilde{E}\phi_i^{(+)}\,\delta(t - t_i) \tag{62}$$

$$\phi^{(l)} = -\Delta_F * \omega^{(l)}, \qquad l \geq 1 \tag{63}$$

where the $\omega^{(l)}$ are given in terms of the previously calculated $\phi^{(l-1)}$ and, for $l > 1$, $\phi^{(l-2)}$. We recall that the Fourier transform of a square-integrable function is also square-integrable. In terms of the Fourier transform $a(\mathbf{k})$ of $g_i^{(+)}(\mathbf{x})$, we can write

$$\phi^{(0)}(x) = (2\pi)^{-3/2} \int d^3k\, (2k_0)^{-1/2} a(\mathbf{k}) \times \exp[-ik_0(t - t_i)] \exp(i\mathbf{k} \cdot \mathbf{x}) \tag{64}$$

which implies that $\|\phi^{(0)}\|$ is finite and independent of t. The Fourier transform of $\phi^{(l)}$ is

$$\tilde{\phi}^{(l)}(\mathbf{k}, t) = \frac{i}{2k_0}\left\{\int_{t_i}^{t} dt'\, \tilde{\omega}^{(l)}(\mathbf{k}, t') \exp[-ik_0(t-t')] + \int_{t}^{t_f} dt'\, \tilde{\omega}^{(l)}(\mathbf{k}, t') \exp[ik_0(t-t')]\right\} \tag{65}$$

whence the L^2 norm of $\tilde{\phi}^{(l)}$ as a function of $\mathbf{k}$ is bounded by virtue of the relation

$$\begin{aligned}\|\tilde{\phi}^{(l)}(\mathbf{k}, t)\|^2 &= \int d^3k\, |\tilde{\phi}^{(l)}(\mathbf{k}, t)|^2 \\ &\leq \int_{t_i}^{t_f} dt' \int \frac{d^3k}{4k_0^2} |\tilde{\omega}^{(l)}(\mathbf{k}, t')|^2 \\ &= \int_{t_i}^{t_f} dt' \left\|\frac{\tilde{\omega}^{(l)}(\mathbf{k}, t')}{2k_0}\right\|^2\end{aligned} \tag{66}$$

Although in the Coulomb gauge the potentials are free of arbitrary (physically meaningless) parts (Marx, 1970c), we use the gauge in which A_0 vanishes to facilitate the proof of convergence of the perturbation series. The fields ϕ in two different gauges are related by the phase factor $\exp(ie\Lambda)$. In this gauge, the source in (58) and (59) reduces to

$$\omega^{(l)} = -2i\nabla \cdot (\mathbf{A}\phi^{(l-1)}) + i\nabla \cdot \mathbf{A}\phi^{(l-1)} - \mathbf{A}^2\phi^{(l-2)}(1-\delta_{1l}) \tag{67}$$

The integrand in (66) is then bounded by

$$\begin{aligned}\|(2k_0)^{-1}\tilde{\omega}^{(l)}(\mathbf{k}, t')\| &= \|(2\tilde{E})^{-1}\omega^{(l)}(\mathbf{x}, t')\| \\ &\leq \sup|\mathbf{A}|\, \|\phi^{(l-1)}\| + \frac{1}{2m} \sup|\nabla \cdot \mathbf{A}|\, \|\phi^{(l-1)}\| \\ &\quad + \sup(\mathbf{A}^2) \|\phi^{(l-2)}\|(1-\delta_{1l})\end{aligned} \tag{68}$$

where the supremum is taken over all space and over the time interval (t_i, t_f). We have used the equality $\|f\| = \|\tilde{f}\|$ and the inequalities $|\mathbf{k}/k_0| < 1$ and $1/k_0 \leq 1/m$. We thus find that

$$\|\phi^{(l)}\| \leq a\|\phi^{(l-1)}\| + b\|\phi^{(l-2)}\|(1-\delta_{1l}) \tag{69}$$

where we now include the L^∞ norm for the t-dependence in the symbol $\|$ and set

$$a = (t_f - t_i) \sup(|\mathbf{A}| + |\nabla \cdot \mathbf{A}|/2m), \qquad b = (t_f - t_i) \sup \mathbf{A}^2/2m \tag{70}$$

Thus, the norms of the terms in the perturbation expansion are bounded above by the terms of the sequence defined by the recursion relation

$$M_l = aM_{l-1} + bM_{l-2}(1-\delta_{1l}), \qquad M_0 = \|\phi^{(0)}\|, \qquad M_l > 0 \tag{71}$$

which has the solution

$$M_1 = aM_0, \qquad M_l = M_1 r^{l-1} \tag{72}$$

$$r = a/2 + [a^2/4 + b]^{1/2} \tag{73}$$

A bound on the norm of a finite sum is

$$\left\| \sum_{l=0}^{n} e^l \phi^{(l)} \right\| \le \sum_{l=0}^{n} e^l M_l = M_0 \left[1 + ea \frac{1-(er)^n}{1-er} \right] \tag{74}$$

whence the perturbation series converges uniformly in t and in L^2 in $\mathbf{x}$ if

$$er < 1, \tag{75}$$

where e is the electric charge and r depends on the electromagnetic potentials.

This concludes the proof of the convergence of the perturbation series defined by the recursion relation (63).

6. SPINOR FIELDS

The bispinor field in an external electromagnetic field obeys the Dirac equation

$$(-i\gamma \cdot D + m)\psi(x) = 0 \tag{76}$$

The conserved current density

$$j_\mu = \bar{\psi}\gamma_\mu\psi \tag{77}$$

has a positive-definite charge density j_0, which gives a charge

$$Q = \int d^3x\, \psi^\dagger \psi = \int d^3p \sum_{\lambda=\pm 1} [|b_\lambda(\mathbf{p}, t)|^2 + |d_\lambda(\mathbf{p}, t)|^2] \tag{78}$$

If we assume that the b_λ and d_λ correspond to the probability amplitudes in momentum space, it is reasonable to combine the helicity states to find the amplitudes in position space. We recall that

$$w_\lambda(p) = [(E+m)/2m]^{1/2}[1 + \boldsymbol{\alpha} \cdot \mathbf{p}/(E+m)]w_\lambda^{(0)}(\hat{p}) \tag{79}$$

where w_λ is a collective designation for u_λ and v_λ and the $w_\lambda^{(0)}$ are constructed from the helicity spinors χ_λ; the precise form of the $w_\lambda^{(0)}$ depends on the choice of γ_μ. We have

$$\gamma_0 u_\lambda^{(0)}(\hat{p}) = u_\lambda^{(0)}(\hat{p}), \qquad \gamma_0 v_\lambda^{(0)}(\hat{p}) = -v_\lambda^{(0)}(\hat{p}) \tag{80}$$

We follow Schröder (1964) and Marx (1968) and define the probability amplitude g (Marx, 1970b) obtained from ψ by a free-field Foldy-Wouthuysen transformation,

$$g(x) = [(\tilde{E}+m)/2\tilde{E}]^{1/2}[1 - i\boldsymbol{\gamma}\cdot\nabla/(\tilde{E}+m)]\psi(x)$$
$$= (2\pi)^{-3/2}\int d^3p \sum_{\lambda}[u_{\lambda}^{(0)}b_{\lambda}(\mathbf{p}, t)\exp(i\mathbf{p}\cdot\mathbf{x})$$
$$+ v_{\lambda}^{(0)}d_{\lambda}(\mathbf{p}, t)\exp(-i\mathbf{p}\cdot\mathbf{x})] \tag{81}$$

and we separate the positive- and negative-frequency parts,

$$g^{(\pm)}(x) = \tfrac{1}{2}(1 \pm \gamma_0)g(x) \tag{82}$$

The use of these amplitudes avoids problems related to *Zitterbewegung*, but the charge

$$Q = \int d^3x\, [g^{(+)\dagger}g^{(+)} + g^{(-)\dagger}g^{(-)}] \tag{83}$$

is the sum of two terms and the probabilistic interpretation based on (47) is no longer valid.

In QFT, the equations are modified by the anticommutation of operators. We can similarly modify the equations of motion (Marx, 1970b) so that the conserved charge is the difference of the two terms, or we can use a many-times QFT (Marx, 1972) to accomplish the same purpose. The solution of the problem of the charged particle in the given electromagnetic field follows the same steps outlined in the previous section.

The free massless spinor field χ forms the conserved current density

$$j_{\mu} = \chi_{\dot{A}}^{*}\sigma_{\mu}^{\dot{A}B}\chi_{B} \tag{84}$$

and the corresponding charge

$$Q = \int d^3p\, (|b|^2 + |d|^2) \tag{85}$$

is also the sum of two positive terms.

7. CONCLUDING REMARKS

We have defined the causal Green function for the Klein–Gordon equation and related distributions, and we have determined the corresponding causal time-boundary conditions.

We have separated the wave function of RQM into positive- and negative-frequency parts to provide the theory with a probabilistic interpretation for particle and antiparticle amplitudes.

We have used these distributions to find the perturbation expansion of the wave function for a charged scalar particle in an external electromagnetic field and we have shown the convergence of this series for appropriately bounded potentials.

The Dirac equation for spin-1/2 particles requires a modification that will make the charge density indefinite before this form of RQM is applicable. This is accomplished in QFT by anticommuting two field operators, and similar modifications can be introduced in RQM.

There is no need in our formulation of RQM for the standard renormalization procedure, since there are no infinite terms in the perturbation expansion.

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